International Journal of Theoretical Physics, Vol. 46, No. 1, January 2007 (© 2006)
DOI: 10.1007/s10773-006-9078-3

Time Flow in a Noncommutative Regime

Leszek Pysiak[1]

Received June 4, 2005; accepted January 26, 2006
Published Online: December 12, 2006

We develop an approach to dynamical and probabilistic properties of the model unifying general relativity and quantum mechanics, initiated in the paper (Heller *et al.* (2005) *International Journal Theoretical Physics* **44**, 671). We construct the von Neumann algebra $\mathcal{M}$ of random operators on a groupoid, which now is not related to a finite group G, but is the pair groupoid of the Lorentzian frame bundle E over spacetime M. We consider the time flow on $\mathcal{M}$ depending on the state ϕ. The state ϕ defining the noncommutative dynamics is assumed to be normal and faithful. Then the pair $(\mathcal{M}, \phi)$ is a noncommutative probabilistic space and ϕ can also be interpreted as an equilibrium thermal state, satisfying the Kubo-Martin-Schwinger condition. We argue that both the "time flow" and thermodynamics have their common roots in the noncommutative unification of dynamics and probability.

KEY WORDS: unification theory; quantum mechanics; noncommutative dynamics; noncommutative probability; random operators; von Neumann algebra.

1. INTRODUCTION

The present paper is a continuation of the paper (Heller *et al.*, 2005a) (to which we shall refer as to the previous paper) in which we have studied probabilistic and dynamical properties of the model unifying general relativity and quantum mechanics. The idea of the model, proposed in (Heller *et al.*, 1997, 2000; Heller and Sasin, 1999) is the following. We consider a principal fibre bundle $\pi_M : E \to M$ over spacetime M with a structural group G, typically the frame bundle with $G = SO_0(3, 1)$. The right action of G on E naturally leads to the construction of the transformation groupoid $\Gamma = E \times G$. On Γ we define a noncommutative algebra $\mathcal{A}$ of smooth functions (compactly supported, if necessary) with convolution as multiplication and, in terms of this algebra and the module of its derivations we develop the differential geometry of this groupoid. The "horizontal component" of the geometry on Γ reproduces the standard spacetime geometry (which is contained in the geometry of the model as a suitable lifting from M). The regular representation $\pi^p : \mathcal{A} \to \mathcal{B}(\mathcal{H}^p)$ of the algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}^p$, for every

[1] Department of Mathematics and Information Science, Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland; e-mail: pysiak@wsisiz.edu.pl.

$p \in E$, leads to a generalized quantum mechanics. In Heller *et al.* (2004, 2005a) we have tested this model for a simpler case when G is a finite group, and in Heller *et al.* (2005b) we have extended this analysis to the case when G is a noncompact group.

In the previous paper, we have demonstrated that every $a \in \mathcal{A}$, if suitably represented in a bundle of Hilbert spaces, is a random operator, an element of a von Neumann algebra $\mathcal{M}$. Moreover, on the strength of the Tomita–Takesaki theorem there exists a one-parameter group of automorphisms of the algebra $\mathcal{M}$, the so-called modular group, which depends on the state ϕ on the algebra $\mathcal{M}$. This one-parameter group can be used to define a noncommutative state depending dynamics. Therefore, the pair $(\mathcal{M}, \phi)$ is a dynamic object. But it turns out that the same object defines a noncommutative probabilistic space with ϕ playing the role of the probability measure (see, e.g., Voiculescu, 1985; Voiculescu *et al.*, 1992). Thus the pair $(\mathcal{M}, \phi)$ is both a dynamic object and a probabilistic object. The goal of the present paper is to go more deeply into this unification of the two so far independent concepts and its physical signification. In the present paper G is not necessarily a finite group. The obvious strategy is to connect dynamic and probabilistic properties of the model with some generalized thermal-like properties. The idea is strengthened by the fact that normal and faithful states on the algebra $\mathcal{M}$ are Kubo–Martin–Schwinger states which, in the standard physics, are interpreted as equilibrium thermal states (Gibbs states). We argue that the "time flow" and the thermodynamics of our world can be deduced from noncommutative probabilistic thermal-like aspects of our model. Since, however, we are dealing with a subtle and not very well explored aspects of noncommutative geometry, our interpretation must be based on hard mathematics. This is why the formal side of the present paper is rather pronounced (with the ample Appendix containing definitions and proofs of some mathematical facts).

To have a self-contained paper, in Section 2, we present basic mathematical aspects of the model. In contrast to the previous paper, we define the groupod Γ as the pair groupoid which makes it more adapted to our present purposes. The algebra $\mathcal{A}$ is defined accordingly. In Section 2, we study the von Neumann algebra $\mathcal{M}$ of random operators, and in Section 3 its dynamic and probabilistic properties. Physical interpretation is discussed in Section 5, and in Section 6 we look for the consequences of the proposed interpretation in quantum and classical limits.

2. BASIC GROUPOID AND ITS CONVOLUTION ALGEBRA

Let E be the principal G-bundle over spacetime M, where G is a semisimple Lie goup. (We can think that G is the Lorentz group and E is the frame bundle over M). Let $\Gamma = E \times_M E$ be a groupoid of the Whitney product of the bundle E with itself, i.e. $E \times_M E = \{(p_1, p_2) : p_1, p_2 \in E, \pi_M(p_1) = \pi_M(p_2)\}$. Let us consider the projections $d, r : \Gamma \to E$ defined by $d(p_1, p_2) = p_1$ and $r(p_1, p_2) = p_2$, and let us distinguish the set $\Gamma^{(2)}$ of composable elements $\Gamma^{(2)} = \{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma : r(\gamma_1) = d(\gamma_2)\}$.

For $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, $\gamma_1 = (p_1, p_2)$, $\gamma_2 = (p_2, p_3)$ we define the groupoid multiplication $\gamma_1 \circ \gamma_2 = (p_1, p_3)$. The groupoid Γ with this multiplication will be called the pair-bundle groupoid.

It is clear that the inverse of γ is always defined and is given by the formula $(p_1, p_2)^{-1} = (p_2, p_1)$. Every element of the form $\gamma = (p, p)$, for any $p \in E$, is called the unit of the groupoid Γ. Thus the space of units can be identified with the set E.

In the papers mentioned in the Introduction we have studied a model unifying general relativity and quantum mechanics in terms of a noncommutative algebra $\mathcal{A}_1$ of functions on a transformation groupoid $\Gamma_1 = E \times G$ with convolution as multiplicaton. Let us observe that the groupoids Γ_1 and Γ are isomorphic. Indeed, the isomorphism $j : \Gamma_1 \to \Gamma$, is given by $j(p, g) = (p, pg)$.

Let us introduce the following fibrations of the space Γ. For $p \in E$ we define the set of elements begining at p

$$\Gamma_p = \{\gamma \in \Gamma : d(\gamma) = p\} = \{(p, p_1) : p_1 \in E, \pi_M(p_1) = \pi_M(p)\}.$$

Analogously, we define the set of elements ending at p

$$\Gamma^p = \{\gamma \in \Gamma : r(\gamma) = p\} = \{(p_1, p) : p_1 \in E, \pi_M(p_1) = \pi_M(p)\}.$$

The sets Γ_p and Γ^p are diffeomorphic to the fiber E_x of the bundle E where $x = \pi_M(p)$.

We consider the algebra $\mathcal{A}$ of smooth complex valued compactly supported functions on Γ with convolution as multiplication. Let us recall that the groupoid convolution is in general defined by the formula

$$(a * b)(\gamma) = \int_{\Gamma_{d(\gamma)}} a(\gamma_1) b\left(\gamma_1^{-1}\gamma\right) d\gamma_1,$$

where $a, b \in \mathcal{A}$, $\gamma, \gamma_1 \in \Gamma$, and $d\gamma_1$ is a measure on the set $\Gamma_{d(\gamma)}$ which is an element of the left Haar measure system of the groupoid Γ (see Paterson, 1999).

In our case of the pair-bundle groupoid, the convolution assumes the following form

$$(a * b)(p_1, p_2) = \int_{E_x} a(p_1, p) b(p, p_2)\, dp,$$

where $\pi_M(p_1) = \pi_M(p_2) = x$

The measure dp on the fiber $\Gamma_{p_1} \simeq E_x$ is related to the Haar measure on G by the formula

$$\int_{E_x} f(p_1, p)\, dp = \int_G f(p_1, p_1 g)\, dg.$$

It is clear that the collection of measures dp on the fibers $\Gamma_p \simeq E_{\pi_M(p)}$ forms a left Haar system of the groupoid Γ.

One can write

$$(a * b)(p_1, p_2) = \int_G a(p_1, p_1 g) b(p_1 g, p_2)\, dg.$$

In the papers (Heller *et al.*, 2004, 2005a,b; Pysiak *et al.*, 2005) we worked with the convolution algebra $\mathcal{A}_1$ of the transformation groupoid Γ_1. From the isomorphism of the groupoids Γ_1 and Γ it follows that the convolution algebras $\mathcal{A}_1$ and $\mathcal{A}$ are isomorphic as well.

3. VON NEUNMANN ALGEBRA OF RANDOM OPERATORS

Now, let us consider the regular representation $\pi_p : \mathcal{A} \to B(\mathcal{H}^p)$ of the algebra $\mathcal{A}$. Here $\mathcal{H}^p$ denotes the Hilbert space of square-integrable functions on Γ^p, $\mathcal{H}^p = L^2(\Gamma^p)$, and $B(\mathcal{H}^p)$ denotes the algebra of bounded operators in $\mathcal{H}^p$.

We have the following formula

$$(\pi_p(a)\psi)(p_1, p_2) = \int_{E_x} a(p_1, p)\psi(p, p_2)\, dp = \int_G a(p_1, p_1 g)\psi(p_1 g, p_2)\, dg.$$

Now, let us define the maps $I_p : L^2(G) \to L^2(\Gamma^p)$, for every $p \in E$, by the formula: $\psi(p_1, p) = \psi^0(p_1/p)$, where $\psi^0 \in L^2(G)$ with p_1 such that $\pi_M(p_1) = \pi_M(p)$, and p_1/p denoting the unique element g of the group G such that $p_1 = pg$. From the definition of the scalar product in $L^2(\Gamma^p)$, which is $(\psi_1, \psi_2) = \int_{E_x} \psi_1(p_1, p)\overline{\psi_2(p_1, p)} dp_1$, one easily deduces that the maps I_p are isomorphisms of the Hilbert spaces $L^2(G)$ and $L^2(\Gamma^p)$.

Now, let us define

$$\tilde{\pi}_p(a) = I_p^{-1} \circ \pi_p(a) \circ I_p.$$

Thus $\tilde{\pi}_p$ is a representation of the algebra $\mathcal{A}$ in the Hilbert space $L^2(G)$, $\tilde{\pi}_p : \mathcal{A} \to B(L^2(G))$.

Proposition 3.1.

$$\tilde{\pi}_{pg_0}(a) = L_{g_o}^{-1} \circ \tilde{\pi}_p(a) \circ L_{g_0}.$$

Proof: First, let us notice that, for $\psi^0 \in L^2(G)$, we have

$$((\pi_p(a) \circ I_p)\psi^0)(p_1, p) = \int_G a(p_1, pg)\psi^0(g)\, dg.$$

By putting $p_1 = pg_1$ we obtain

$$((\tilde{\pi}_p(a)\psi^0)(g_1) = \int_G a(pg_1, pg)\psi^0(g)\, dg$$

which leads to

$$\left(L_{g_0}^{-1} \circ \tilde{\pi}_p(a) \circ L_{g_0}\right)\psi^0(g_1) = \int_G a(pg_0g_1, pg)\psi^0\left(g_0^{-1}g\right) dg$$

$$= \int_G a(pg_0g_1, pg_0h)\psi^0(h)\, dh = (\tilde{\pi}_{pg_0}\psi^0)(g_1)$$

where in the second equality we have used the Haar measure invariance. □

Let us recall some facts related to random operators. By definition, *the random operator* r on the groupoid Γ is a collection of operators $(r_p)_{p\in E}$, i.e., a function

$$r : E \to \bigsqcup_{p\in E} \mathcal{B}(\mathcal{H}^p),$$

such that

1. for every $\phi, \psi \in L^2(G)$ the complex-valued function given by
$$E \ni p \mapsto (r(p)I_p(\phi), I_p(\psi)) \in \mathbf{C}$$
is measurable with respect to the standard manifold measure on E;
2. given the norm of r by
$$||r|| = \operatorname{ess\,sup}||r(p)||_p$$
one has $||r|| < \infty$. Here "ess sup" is taken with respect to $p \in E$ and $||r(p)||_p$ denotes the usual operator norm in the space $B(\mathcal{H}^P)$.

Let us observe that, for every $a \in \mathcal{A}$, the family of operators $(\pi_p(a))_{p\in E}$ forms a random operator. Indeed, the first condition is satisfied by the above formula for the operator $\tilde{\pi}_p(a)$. Let us denote $a_p(g_1, g_2) = a(pg_1, pg_2)$. It can be easily seen that

$$||\pi_p(a)|| = ||\tilde{\pi}_p(a)|| \leq ||a_p||_{L^2(G\times G)} \leq \sup|a| \cdot C$$

where C is a positive constant. Thus the family $\pi_p(a)$ satisfies the second condition of the above definition as well.

Let $\tilde{r}$ denote the realization of a random operator r in the space $L^2(G)$ given by $\tilde{r}(p) = I_p^{-1} \circ r(p) \circ I_p$. Now, we can introduce the concept of *covariant random operator* on the groupoid Γ. By definition, it will be a random operator r which satisfies the relation

$$\tilde{r}(pg_0) = L_{g_o}^{-1} \circ \tilde{r}(p) \circ L_{g_0}.$$

Proposition 3.1 implies that the family $(\pi_p(a))_{p\in E}$ is a covariant random operator. The operator of such a form will be denoted by r_a.

Let $\mathcal{M}$ be the $*$-algebra of equivalence classes of covariant random operators on Γ (modulo equality almost everywhere of operator-valued functions defined on E). It is clear that algebraic operations are defined in the pointwise way on representatives in the usual manner and the class of the operator $1(p) = Id_{\mathcal{H}^p}$ is the unit.

Let us observe that we have the following isomorphisms

$$\mathcal{M} \simeq L^\infty_G(E, B(L^2(G))) \simeq L^\infty(M, B(L^2(G))),$$

(see Appendix B) and we conclude that $\mathcal{M}$ has the structure of a von Neumann algebra. This algebra will be called *the von Neumann algebra of the groupoid* Γ. The second isomorphism can be understood as a generalization of the matrix representation of the groupoid algebra constructed in the finite case in Heller *et al.* (2004).

Let us denote by $\mathcal{M}_0$ the subalgebra of $\mathcal{M}$ formed by operators r_a, $a \in \mathcal{A}$. In Appendix B we show that $\mathcal{M}_0$ is σ-weakly dense subalgebra of $\mathcal{M}$.

4. STATE-DEPENDENT EVOLUTION OF RANDOM OPERATORS

To describe the evolution of random operators we can make use of the Tomita–Takesaki theorem (Sunder, 1987; Connes, 1994). We assume that ϕ is a normal faithful state on $\mathcal{M}$. We know (see Appendix B) that $\phi(r) = Tr(r\hat{\rho})$ where $\hat{\rho}$ is a trace-class operator ($\hat{\rho} \in \mathcal{M}_*$) called the density operator and $r \in \mathcal{M}$. It is easily seen that ϕ can be written in the form

$$\phi(r) = Tr(\hat{\rho}r) = \int_M tr(\hat{\rho}(p)r(p))\, d\mu_M(x)$$

where $\hat{\rho}(p)$ is a positive injective operator of trace-class in $B(\mathcal{H}^p)$, for every $p \in E$, and it is G-covariant. The above integral is taken with respect to the Lebesgue type manifold measure on M and it is well defined since $tr(\hat{\rho}(p)r(p))$ depends only on $x = \pi_M(p)$. With these conditions the state ϕ satisfies all conditions of the Tomita–Takesaki theorem. There exists a basis $\{e_i\}$ in $\mathcal{H}^p$ such that $\hat{\rho}(p)e_i = \lambda_i(p)e_i$, $\lambda_i > 0$. We also have

$$\sum_{i=0}^{\infty} \lambda_i(p) = \lambda(p) < \infty$$

for almost every $p \in E$, and $\lambda(\cdot) \in L^1(E)$ with

$$\int_M \lambda(p)\, d\mu_M(x) = 1,$$

and the integral is well defined. If we define the Hamiltonian $H_p^\phi = \mathrm{Log}\,\hat{\rho}(p)$, we have $H_p^\phi(p)e_i = (\log\lambda_i(p))e_i$. We thus can write the state dependent evolution of random operators $r \in \mathcal{M}$ as

$$\sigma_s^\phi(r(p)) = e^{isH_p^\phi} r(p) e^{-isH_p^\phi}.$$

Here we have denoted by σ_s^ϕ an automorphism of $\mathcal{M}$, and $\{\sigma_s^\phi\}_{s\in\mathbf{R}}$ forms a one-parameter group of automorphisms (the "time evolution," see below) of random operators with respect to the parameter $s \in \mathbf{R}$ (the so-called modular group). After differentiating the above equation (and inserting $\hbar$ for the correspondence with the known quantum mechanical case) it can be rewritten as

$$i\hbar \frac{d}{ds}|_{s=0}\sigma_s^\phi(r(p)) = \left[r(p), H_p^\phi\right].$$

This equation, describing the evolution of random operators with respect to the parameter $s \in \mathbf{R}$, is a generalization of the Heisenberg equation of the usual quantum mechanics. The essential difference is that now the dynamics depends on the state ϕ. How should we interpret this fact? As well known, von Neumann algebras are regarded as a noncommutative counterpart of the measure theory, and the pair $(\mathcal{M}, \phi)$, where ϕ is a normal and faithful state on $\mathcal{M}$, is defined to be a noncommutative probability space (see, for instance Voiculescu *et al.*, 1992). In contrast to the commutative case, where there is only one interesting measure (the Lebesgue measure), in the noncommutative case there is a great richness of measures. If we take into account the fact that the state ϕ is also a probability measure, it seems natural that the evolution of random operators depends on probability measure that has been used.

The question arises: what is the relation between two dynamics corresponding to two different states (or measures) ϕ and ψ? We can answer this question by applying the construction based on the Connes–Nicodym–Radon theorem (Sunder, 1987, p. 74, see also Connes, 1994). This is done in the following way. Let $\mathcal{U} = \{u \in \mathcal{M} : uu^* = u^*u = 1\}$ denote the unitary group of the algebra $\mathcal{M}$, and $\mathrm{Aut}\,\mathcal{M}$ the group of all automorphisms of $\mathcal{M}$. An automorphism $\lambda \in \mathrm{Aut}\mathcal{M}$ is called *inner* if there exists an element $u \in \mathcal{U}$, such that

$$\lambda(b) = ubu^*$$

for every $b \in \mathcal{M}$. Two automorphisms λ_1 and λ_2 are said to be *inner equivalent* if

$$\lambda_1(b) = u\lambda_2(b)u^*,$$

for every $b \in \mathcal{M}$.

In Appendix B we prove that $\mathcal{M}$ is semifinite. But the Dixmier–Takesaki theorem (see Connes, 1994) says that if $\mathcal{M}$ is semidefinite then two groups of modular automorphisms σ_s^ϕ and σ_s^ψ are inner equivalent, i.e., for every s there exists U_s in $\mathcal{U}$ such that

$$U_s \sigma_s^{\phi}(r(p)) U_s^* = \sigma_s^{\psi}(r(p))$$

This fact can be interpreted by stating that two dynamics differ only on some gauge transformation which can also be used to define the equivalence of the corresponding measure spaces $(\mathcal{M}, \phi)$ and $(\mathcal{M}, \psi)$.

Another important property of a normal faithful state ϕ on $\mathcal{M}$ is that it satisfies the Kubo–Martin–Schwinger (KMS) condition with respect to the group of automorphisms $\{\sigma_s^{\phi}\}$. (See Appendix C). This implies that the dynamics $\{\sigma_s^{\phi}\}$ satisfies the condition: $\phi \circ \sigma_s^{\phi} = \phi$. KMS states are usually interpreted as equilibrium thermal states (the generalized Gibbs states).

5. INTERPRETATION

Mathematical facts established in the preceding sections have far-reaching consequences for our model. To disclose them is a goal of the present section. In pursuing this goal we adapt the deep analysis made by Connes and Rovelli (1994) to the context of our model.

Let us first consider a classical Hamiltonian system. Its phase space is denoted by Σ, and its observables are elements of $C^{\infty}(\Sigma)$. The Hamiltonian H of the system defines a flow

$$\alpha_t^{\Sigma} : \Sigma \to \Sigma,$$

$t \in \mathbf{R}$, which can be regarded as a counterpart of the Schrödinger picture of quantum mechanics. The same dynamics can be presented with the help of the one-parameter group of automorphisms of the algebra $C^{\infty}(\Sigma)$ in the following way

$$(\alpha_t f)(x) = f(\alpha_t^{\Sigma} x),$$

for $f \in C^{\infty}(\Sigma)$, $x \in \Sigma$, which is clearly the counterpart of the Heisenberg picture.

In our model there is no "external time parameter," and consequently we cannot hope to implement the "Schrödinger picture," but we can generalize the "Heisenberg picture" with the help of the one-parameter group of automorphisms of the algebra of random operators. We thus make the assumption that in our model the "time flow" is defined by the modular group $\{\sigma_s^{\phi}\}_{s \in \mathbf{R}}$. This one-parameter group depends on ϕ which is interpreted both as a state and as a probability measure. Since we are dealing with random operators it is natural that their dynamic depends on the probability definition.

We meet similar situations, where dynamics involves probability, in classical statistical mechanics and in quantum field theories with thermal field fluctuations. Basing on these analogies we can guess that the dynamics of random operators in our model is also connected with "thermal-like fluctuations" with respect to a given probability measure ϕ. In classical statistical mechanics the equilibrium thermal states (Gibbs states) are given by the formula

$$\rho \sim e^{-\beta H}.$$

where $\beta = 1/kT$, and k being the Boltzmann constant and T the absolute temperature. Therefore, the time flow α_t of classical statistical mechanics, in a given inverse temperature β, can be recovered from both the Gibbs state ρ and the Hamiltonian H. In our model there is no unique Hamiltonian, thus the only way to recover the above mentioned analogies is to use the generalized Gibbs states, i.e. normal, faithful states ϕ on the von Neumann algebra $\mathcal{M}$ which, as we have seen, are just KMS states.

In classical statistical mechanics, a given state is not represented as a point in the phase space but as a measure on it. With the help of this measure the mean values of observables are defined. If the system finds itself in a constant temperature, the corresponding measure is called Gibbs canonical ensemble. In our case, the role of a measure is played by KMS states ϕ. By following the analogy with classical statistical mechanics we could say that we have an enseble of probability spaces $(\mathcal{M}, \phi)_{\phi \in F}$ where F denotes the collection of normal and faithful states on $\mathcal{M}$, and the dynamics of the system is given by modular groups σ_s^{ϕ}. Our claim that both the "time flow" and thermodynamics have their common roots in the noncommutative unification of dynamics and probability seems to be fully justified.

6. QUANTUM MECHANICAL LIMIT

Let us denote by $\mathcal{M}_{0H}$ the subset of Hermitian random operators in $\mathcal{M}_0$. Let us further assume that $r_a \in \mathcal{M}_{0H}$. Since π_p is *-representation of the algebra $\mathcal{A}$, then $r_a = \pi_p(a)$ for a Hermitian element of $\mathcal{A}$ ($a \in \mathcal{A}$ is Hermitian if $a(p_1, p_2) = \overline{a}(p_2, p_1)$ for every $p_1, p_2 \in E$). The operator r_a is compact for $a \in \mathcal{A}$ as an integral operator in the space $L^2(\Gamma^p)$. On the strength of the spectral theorem for Hermitian compact operators in a separable Hilbert space, there exists in $\mathcal{H}^p$ an orthonormal countable basis composed of eigenvectors $\{\psi_i\}_{i \in I}$ of the operator $r_a(p)$. Therefore, we can write its eigenvalue equation as

$$r_a(p)\psi_i(p) = \lambda_i(p)\psi_i(p)$$

for every $p \in E$. Here $\lambda_i : E \to \mathbf{R}$ is an eigenfunction (not an eigenvalue) of $r_a(p)$. In this sense the above equation should be called the eigenfunction equation rather than the eigenvalue equation. However, when a measurement is performed at a given place $x \in M$ (i.e., in a local frame $p \in \pi_M^{-1}(x)$), then the eigenfunction $\lambda_i(p)$ collapses to the eigenvalue $\lambda_i(p)$. This also singles out the isomorphism $I_p^{-1} : \mathcal{H}^p \to L^2(G)$ which automatically reproduces the usual quantum mechanics (on the group G) (for details see: Heller *et al.*, 2005b).

We could summarize the above considerations by saying that it is the very act of measurement that cuts off the usual quantum mechanics from a larger noncommutative structure. This fact could be responsible for many peculiarities of this physical theory.

We should also notice that the above interpretation of the origin of the time flow and thermodynamics, when applied to quantum systems in thermal equilibrium, reduces to the usual approach. Indeed, the state of a quantum system (with a finite number of degrees of freedom) is described by the Gibbs density matrix

$$\rho = Ne^{-\beta H}$$

where H is the Hamiltonian of the system and $N = (tr(\exp[-\beta H])^{-1}$. The modular group corresponding to the flow generated by the Hamiltonian H is given by

$$\sigma_s A = e^{i\beta tH} A e^{-i\beta tH}$$

where A is an element of a suitable algebra of observables. To show this we must take into account the uniqueness of the modular group σ_t^ϕ for a KMS state ϕ (Appendix C, Proposition C.1, condition 3) and perform the time rescalling $t \to \beta t$ (for a full discussion see Connes and Rovelli, 1994, Section 4).

APPENDIX A: GLOSSARY–VON NEUMANN ALGEBRA, ITS STATES AND WEIGHTS

- A *-subalgebra $\mathcal{M}$ of the algebra $B(H)$ of bounded operators on a Hilbert space H is called *the von Neumann algebra* if it satisfies one of the following equivalent conditions:
  1. If we denote by $\mathcal{M}''$ the double commutant of $\mathcal{M}$, where the commutant $\mathcal{M}' = \{A \in B(H) : AB = BA \text{ for all } B \in \mathcal{M}\}$ and $\mathcal{M}'' = (\mathcal{M}')'$, then we have
  $$\mathcal{M} = \mathcal{M}''.$$
  2. $\mathcal{M}$ is weakly closed in $B(H)$.
  3. $\mathcal{M}$ is σ-weakly closed in $B(H)$.

  (Sunder, 1987, Introduction). The statement of equivalence of the above conditions is called the von Neumann double commutant theorem.
- A linear functional $\varphi : \mathcal{M} \to \mathbf{C}$ is a *state* on $\mathcal{M}$ if $\varphi(r) \geq 0$ for every $r \in \mathcal{M}_+$, where $\mathcal{M}_+ = \{x \cdot x^* : x \in \mathcal{M}\}$ is the subset of positive elements of $\mathcal{M}$, and $\varphi(1) = 1$.
- A functional $\varphi : \mathcal{M}_+ \to [0, \infty]$ is a *weight* if φ is additive, i.e., $\varphi(x + y) = \varphi(x) + \varphi(y)$, and positively homogeneous, i.e., $\varphi(\lambda x) = \lambda\varphi(x)$, for $\mathbf{R} \ni \lambda \geq 0$, $x, y \in \mathcal{M}$. We additionally assume that $\lambda + \infty = \infty$, $\lambda \cdot \infty = \infty$

if $\lambda \neq 0$, and $\lambda \cdot \infty = 0$ if $\lambda = 0$. Let us notice that every state defines a weight.

- A weight φ is *faithful* if for $r \in \mathcal{M}_+$ one has: $\varphi(r) = 0 \Rightarrow r = 0$.
- A weight φ is *normal* if $\varphi(r) = \sup \varphi(r_i)$ provided r is the supremum of a monotonically increasing net $\{r_i\}$ in the collection of positive operators in $\mathcal{M}$. (The sufficient and necessary condition for a weight φ to be *normal* is: $\varphi(x) = \sum_i \varphi_i$ for a family $\{\varphi_i\}$ of normal states. (Bratteli and Robinson, 1987, Theorem 2.7.11).
- A state φ is *normal* if there exists a density matrix $\hat{\varrho}$, i. e. a positive trace-class operator $\hat{\varrho}$ with $Tr\hat{\varrho} = 1$ such that $\varphi(r) = Tr(\hat{\varrho}r)$. (Bratteli and Robinson, 1987, Theorem 2.4.21).
- Let us define: $D_\varphi := \{x \in \mathcal{M}_+ : \varphi(x) < \infty\}$ and $\mathcal{M}_\varphi := \mathrm{Span}_{\mathbf{C}}(D_\varphi)$, i.e. $\mathcal{M}_\varphi$ is the space of $\mathbf{C}$-linear combinations of elements of D_φ. A weight φ is *semifinte* if $\mathcal{M}_\varphi$ is σ-weakly dense in $\mathcal{M}$. (Sunder, 1987, p. 56).
- A weight φ is a *trace* if $\varphi(r^* \cdot r) = \varphi(r \cdot r^*)$, for every $r \in \mathcal{M}$.
- A von Neumann algebra $\mathcal{M}$ is *semifinite* if there exists a faithful, normal, semifinite weight φ on $\mathcal{M}$ which is a trace.

APPENDIX B: PROPERTIES OF THE ALGEBRA $\mathcal{M}$ OF COVARIANT RANDOM OPERATORS

Theorem B.1. *Let $\mathcal{M}$ be the algebra of covariant random operators on Γ. Then*

1. *We have *-algebraic isomorphism*

$$\mathcal{M} \simeq L^\infty_G(E, B(L^2(G))) \simeq L^\infty(M, B(L^2(G))),$$

2. *$\mathcal{M}$ is a von Neumann algebra of operators on the Hilbert space $H = L^2(M, L^2(G))$.*

Proof: As we have seen in Section 3, a covariant random operator r has, for every $p \in E$, a realization $\tilde{r}(p)$ in the Hilbert space $L^2(G)$ which satisfies the condition $\tilde{r}(pg_0) = L_{g_o}^{-1} \circ \tilde{r}(p) \circ L_{g_0}$. We shall denote the algebra of classes of such random operators by $L^\infty_G(E, B(L^2(G)))$. Let s be a measurable section of the bundle $\pi_M : E \to M$, i.e. $s : M \to E$ such that $\pi_M(s(x)) = x$. Let us define, for $x \in M$, the operator $\tilde{\tilde{r}}(x) = \tilde{r}(s(x))$ acting in the Hilbert space $L^2(G)$. In this manner we obtain a measurable operator-valued function $M \ni x \to \tilde{\tilde{r}}(x)$, which is bounded with respect to the norm $||\tilde{\tilde{r}}|| = \operatorname{ess\,sup}||\tilde{\tilde{r}}(x)||$, and the function $\tilde{\tilde{r}}$ defines an element of the algebra $L^\infty(M, B(L^2(G)))$. Let us recall that the action of the group G on the bundle E is free and transitive on the fibers. This fact together with the covariance condition of the operator $\tilde{r}$ implies that the mapping

$L^{\infty}_{G}(E, B(L^2(G)) \to L^{\infty}(M, B(L^2(G)))$, given by $\tilde{r} \mapsto \tilde{\tilde{r}}$ is a bijection. It is clear that it is an isomorphism of algebras. This ends the proof of part 1.

To prove the second part, let us consider the representation of the algebra $\mathcal{M} \simeq L^{\infty}(M, B(L^2(G)))$ in the Hilbert space $H \simeq L^2(M, L^2(G))$ given by $(A\psi)(x) = A(x)\psi(x)$, where $A \in \mathcal{M}$, $\psi \in H$, $x \in M$.

It is well known that this algebra is a von Neumann algebra (see Connes, 1994, ch. V, p. 452) and it can be represented as a direct integral of factors. □

Let us consider the following trace in the von Neumann algebra $\mathcal{M} \simeq L^{\infty}(M, B(L^2(G)))$:

$$Tr(A) = \int_M tr(A(x))\, d\mu(x),$$

where $A \in \mathcal{M}$ and tr denotes the canonical trace in the Hilbert space $L^2(G)$, i.e. $tr A(x) = \sum_i (A(x)\psi_i, \psi_i)$, with (ψ_i) being an orthonormal basis in $L^2(G)$. Tr is a linear functional on $\mathcal{M}$, and if restricted to $\mathcal{M}_+$, is a weight. This weight is faithful. Indeed, for $A = A_1 \cdot A_1^*$ with $A_1 \in \mathcal{M}$, the condition $TrA = 0$ means $tr(A_1(x) \cdot A_1^*(x)) = 0$ for almost every $x \in M$. But this leads to the conclusion that $A = 0$.

Let us denote by $\mathcal{M}_*$ the space $\mathcal{M}_\phi$ introduced in Appendix A for the weight $\phi = Tr$. This space has the structure of a Banach space with the norm $Tr|A|$, and is called the predual of $\mathcal{M}$. We shall call the elements of $\mathcal{M}_*$ the trace-class operators. It is known that $\mathcal{M}$ is the dual space of $\mathcal{M}_*$ if the duality is given by

$$\mathcal{M} \times \mathcal{M}_* \ni (A, \hat{\rho}) = Tr(A\hat{\rho})$$

(See Bratteli and Robinson, 1979, Section 2.4.3).

The σ-weak topology on $\mathcal{M}$ is the weak *-topology, inherited by $\mathcal{M}$ on the strength of the above duality (See Sunder, 1987, p. 10). Let us observe that normal states are precisely σ-weakly continuous states and are given by $\phi(A) = Tr(A\hat{\rho})$ for some $\hat{\rho} \in \mathcal{M}_* \cap \mathcal{M}_+$ and $Tr(\hat{\rho}) = 1$.

Now, we can show that Tr is a normal weight on $\mathcal{M}$. Since the measure on M is σ-finite, one has a compactly supported sequence of functions $f_i \geq 0$, $f_i \in L^1(M)$ and $\sum f_i = 1$. Let P_{ψ_j} denote the projection onto the basis vector ψ_j in $L^2(G)$. It is clear that $\hat{\rho}_{ij} = f_i P_{\psi_j}$ are trace-class operators and, by defining $\phi_{ij}(A) = Tr(\hat{\rho}_{ij} A)$, we obtain $Tr(A) = \sum \phi_{ij}(A)$. This proves that Tr is a normal weight.

Let ρ be a function defined on $M \times G \times G$, such that $\rho(x, g_1, g_2) \geq 0$, $\rho(x, g_1, g_2) = \rho(x, g_2, g_1)$ for every $x \in M, g_1, g_2 \in M$ and $\rho \in L^1(M \times G \times G)$. We assume that ρ has its L^1 - norm equal to one. Such a function ρ will be called the density function of a given state.

Proposition B.2. *Normal states on $\mathcal{M}$ restricted to $\mathcal{M}_0$ are integral functionals of the form*

$$\phi(A) = \int_{G\times G\times M} \tilde{a}(x, g_1, g_2)\rho(x, g_1, g_2)\, dg_1 dg_2\, d\mu(x)$$

where $A \in \mathcal{M}_0$, $A = (\pi_p(a))_{p\in E}$, is the random operator corresponding to the function $a \in \mathcal{A}$, $\tilde{a}(x, g_1, g_2) = a_{s(x)}(g_1, g_2) = a(s(x)g_1, s(x)g_2)$, and ρ is the density function.

We omit the proof which is a standard calculation using the properties of the orthonormal basis of the Hilbert space $L^2(G)$.

We end this Section with the following

Theorem B.3.

1. *The subalgebra $\mathcal{M}_0$ is σ-weakly dense in $\mathcal{M}$.*
2. *The von Neumann algebra $\mathcal{M}$ is semifinite.*

Proof: We have seen that for $A \in \mathcal{M}_0$, $A(x)$ is an integral operator in $L^2(G)$, thus it is compact. This implies that A can be approximated by finite-rank operator valued functions on M. But this means that $\mathcal{M}_0$ is σ-weakly dense in $\mathcal{M}$. The second part is now obvious since $\mathcal{M}_0 \subset \mathcal{M}_*$, and we obtain the semifinitness of the weight *Tr*. But we have already seen that it is normal faithful trace. □

APPENDIX C: KMS STATES ON THE VON NEUMANN ALGEBRA $\mathcal{M}$

Definition C.1. *A state ϕ on the von Neumann algebra $\mathcal{M}$ is said to satisfy the Kubo–Martin–Schwinger condition with respect to a one-parameter group $\{\sigma_s : s \in \mathbf{R}\}$ of automorphisms of $\mathcal{M}$ if, for each $A, B \in \mathcal{M}$ there exists a bounded continuos function $F : \{z \in \mathbf{C} : 0 \le Imz \le 1\} \to \mathbf{C}$, which is analytic in the interior of the strip, and satisfies.*

$$F(s + i) = \phi(A\sigma_s(B))$$

and

$$F(s) = \phi(\sigma_s(B)A),$$

for all $s \in \mathbf{R}$. For brevity, we shall call such a ϕ the KMS state.

Proposition C.2. *(See Sunder, 1987)*

1. *If ϕ is a KMS state with respect to a one-parameter group $\{\sigma_s : s \in \mathbf{R}\}$ then ϕ is σ_s-invariant, i.e.*

$$\phi \circ \sigma_s = \phi$$

2. *Let ϕ be a normal faithful state and $\{\sigma_t^{\phi} : t \in \mathbf{R}\}$—the one-parameter group given by the Tomita-Takesaki theorem. Then ϕ satisfies the KMS condition with respect to $\{\sigma_t^{\phi} : t \in \mathbf{R}\}$, and thus $\phi \circ \sigma_t^{\phi} = \phi$.*
3. *The one-parameter group $\{\sigma_t^{\phi} : t \in \mathbf{R}\}$, for which ϕ is a KMS state is uniquely determined.*

We end this subsection by checking directly that a normal faithful state ϕ on our algebra $\mathcal{M}$ of random operators satisfies the KMS condition with respect to σ_s^{ϕ}.

As we have seen $\phi(A) = Tr(A\hat{\rho})$ for some $\hat{\rho} \in \mathcal{M}_* \cap \mathcal{M}_+$, and $Tr(\hat{\rho}) = 1$. Let us define, for $z \in \mathbf{C}$,

$$\sigma_z^{\phi}(A(p)) = e^{izII_p^{\phi}} A(p) e^{-izII_p^{\phi}},$$

and

$$F(z) = \phi(\sigma_z^{\phi}(B)A).$$

Then $F(z)$ is analytic in the strip $\{z \in \mathbf{C} : 0 < Imz < 1\}$ and continuous on its closure. It follows that $F(s) = \phi(\sigma_s^{\phi}(B)A)$ and $F(s+i) = \phi(e^{i(s+i)II_p^{\phi}} B e^{-i(s+i)II_p^{\phi}} A) = Tr(e^{-II_p^{\phi}} e^{isII_p^{\phi}} B e^{-isII_p^{\phi}} e^{H_p^{\phi}} A\hat{\rho})$.

Now, the fact that $\hat{\rho} = e^{H_p^{\phi}}$ and the property of trace imply that $F(s+i) = \phi(A\sigma_s^{\phi}(B))$.

REFERENCES

Bratteli, O. and Robinson, D. W. (1979). *Operator Algebras and Quantum Statistical Mechanics I,* Springer, New York.

Connes, A. (1994). *Noncommutative Geometry,* Academic Press, New York, London.

Connes, A. and Rovelli, C. (1994). *Classical and Quantum Gravity* **11**, 2899.

Heller, M. and Sasin, W. (1999). *International Journal Theoretical Physics* **38**, 1619.

Heller, M., Sasin, W., and Lambert, D. (1997). *Journal of Mathematical Physics* **38**, 5840.

Heller, M., Sasin, W., and Odrzygóźdź, Z. (2000). *Journal of Mathematical Physics* **41**, 5168.

Heller, M., Odrzygóźdź, Z., Pysiak, L., and Sasin, W. (2004). *General Relativity and Gravitation* **36**, 111.

Heller, M., Pysiak, L., and Sasin, W. (2005a). *International Journal of Theoretical Physics* **44**, 671.

Heller, M., Pysiak, L., and Sasin, W. (2005b). Journal of Mathematical Physics **46**, 122501.

Paterson, A. L. (1999). *Groupoids, Inverse Semigroups and Their Operator Algebras*, Birkhäuser, Boston Mass.

Pysiak, L. (2004). *Demonstratio Mathematica* **37**, 661.

Pysiak, L., Heller, M., Odrzygóźdź Z., and Sasin, W. (2005). *General Relativity and Gravitation* **37**, 541.

Rovelli, C. (1993). *Classical Quantum Gravity* **10**, 1549.

Sunder, V. S. (1987). *An Invitation to von Neumann Algebras*, Springer, New York.
Voiculecsu, D. V. (1985). In: *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Mathematics, vol. 1132, Springer, Berlin.
Voiculescu, D. V., Dykema, K., and Nica, A. (1992). *Free Random Variables*, CRM Monograph Series No 1. American Mathematical Society, Providence.